

Fractal Structure of Loop Quantum Gravity

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In this paper we have calculated the spectral dimension of loop quantum gravity (LQG) using simple arguments coming from the area spectrum at different length scales. We have obtained that the spectral dimension of the spatial section runs from 2 to 3, across a 1.5 phase, when the energy of a probe scalar field decreases from high to low energy. We have calculated the spectral dimension of the space-time also using results from spin-foam models, obtaining a 2-dimensional effective manifold at high energy. Our result is consistent with other two approach to non perturbative quantum gravity: *causal dynamical triangulation* and *asymptotic safety quantum gravity*.

In the past years many approaches to quantum gravity have studied the fractal properties of the space-time. In particular in *causal dynamical triangulation* (CDT) [1] and *asymptotic safety quantum gravity* (ASQG) [2] a fractal analysis of the space-time gives a two dimensional effective manifold at high energy. In the two approaches the spectral dimension is $\mathcal{D}_s = 2$ at small scales and $\mathcal{D}_s = 4$ at large scales. Recently the previous ideas have been applied in the context of *non commutativity* to a quantum sphere and κ -Minkowski [3]. In particular for the second group the author found a space-time spectral dimension $\mathcal{D}_s = 3$ at high energy. The spectral analysis is a useful tool to understand the *effective form* of the space at small and large scales. We think the fractal analysis could be also useful to predict the behavior of the 2-points and n -points functions at small scales and to attack the singularity problems of general relativity in a full theory of quantum gravity.

In this paper we apply to *loop quantum gravity* (LQG) [4] [7] the analysis developed in the context of ASQG by O. Lauscher and M. Reuter [2]. In the context of LQG we consider the spatial section which is a $3d$ manifold and we extract the energy scaling of the metric from the area spectrum. We consider the $SU(2)$ representations, which appear in the area spectrum, as continuum variables. This is a strong approximation but the result will be consistent with the well-known interpretation given in [4]. We apply the same analysis to the space-time using the area spectrum that is suggested by the spin-foam models [5]. In the space-time case the result will be consistent with the spectral dimension calculated in the different approach to *non-perturbative quantum gravity* [1], [2].

The paper is organized as follows. In the first section we extract information about the scaling property of the $3d$ spatial section metric from the area spectrum of LQG. The same analysis in the context of spin-foam models gives the scaling properties of the metric in $4d$. In the second section we give a short review of the spectral dimension in diffusion processes. In the third section we calculate explicitly the spectral dimension of the spatial section in LQG and the space-time dimension using the area spectrum from spin-foam models [5].

a. Scaling of the metric One of the strongest results of LQG is the quantization of the area, volume and re-

cently length operators [8]. In this section we recall the area spectrum and we deduce from that the energy scaling of the $3d$ -metric of the spatial section. The area spectrum on a spin-network state, $|\gamma; j_e, \iota_n\rangle$, without edges and nodes on the surface \mathcal{S} we are considering is

$$\hat{A}_{\mathcal{S}}|\gamma; j_e, \iota_n\rangle = 8\pi\gamma G_N \hbar \sum_{p \cap \mathcal{S}} \sqrt{j_p(j_p + 1)} |\gamma; j_e, \iota_n\rangle, \quad (1)$$

where j_p are the representations on the edges that cross the surface \mathcal{S} . We will restrict to the case when a single edge crosses the surface. Using (1) we can calculate the relation between the area operator average for two different states of two different $SU(2)$ representations, j and j_0 ,

$$\langle \gamma; j | \hat{A} | \gamma; j \rangle = \frac{l_P^2 \sqrt{j(j+1)}}{l_P^2 \sqrt{j_0(j_0+1)}} \langle \gamma_0; j_0 | \hat{A} | \gamma_0; j_0 \rangle. \quad (2)$$

We can introduce the length square defined by $l^2 = l_P^2 j$ and the infrared length square $l_0^2 = l_P^2 j_0$. Using this definition we obtain the scaling properties of the area's eigenvalues. If $\langle \hat{A}_l \rangle$ is the area average at the scale l and $\langle \hat{A}_{l_0} \rangle$ is the area average at the scale l_0 (with $l \leq l_0$) then we obtain the scaling relation

$$\langle \hat{A}_l \rangle = \frac{\sqrt{l^2(l^2 + l_P^2)}}{\sqrt{l_0^2(l_0^2 + l_P^2)}} \langle \hat{A}_{l_0} \rangle. \quad (3)$$

The classical area operator can be related to the spatial metric g_{ab} in the following way. The classical area operator can be expressed in terms of the density triad operator,

$$A_{\mathcal{S}} = \int_{\mathcal{S}} \sqrt{n_a E_i^a n_b E_i^b} d^2\sigma, \quad (4)$$

and the density triad is related to the three dimensional triad by $e_i^a = E_i^a / \sqrt{\det E}$ and $\sqrt{\det E} = \det e$. If we rescale the area operator by a factor Q^2 , $A \rightarrow A' = Q^2 A$, consequently the density triad scales by the same quantities, $E_i^a \rightarrow E_i^{a'} = Q^2 E_i^a$. The triad instead, using the above relation, scales as $e_i^a \rightarrow e_i^{a'} = Q^{-1} e_i^a$ and the inverse $e_a^i \rightarrow e_a^{i'} = Q e_a^i$. The metric on the spatial section is related to the triad by $g_{ab} = e_a^i e_b^j \delta_{ij}$ and then it scales

as $g_{ab} \rightarrow g'_{ab} = \mathcal{Q}^2 g_{ab}$, or in other words the metric scales as the area operator.

Using (3) and (4) we obtain the following scaling for the metric

$$\langle \hat{g}_{ab} \rangle_l = \left[\frac{l^2(l^2 + l_P^2)}{l_0^2(l_0^2 + l_P^2)} \right]^{\frac{1}{2}} \langle \hat{g}_{ab} \rangle_{l_0}. \quad (5)$$

If we want to observe the spatial section with a microscope of resolution l we must use a probe field of momentum $k \sim 1/l$. The scaling property of the metric in terms of k can be obtained replacing: $l \sim 1/k$, $l_0 \sim 1/k_0$ and $l_P \sim 1/E_P$. Where k_0 is an infrared energy cutoff and E_P is the Planck energy. The scaling of the metric in the momentum space is

$$\langle \hat{g}_{ab} \rangle_k = \left[\frac{k_0^4(k^2 + E_P^2)}{k^4(k_0^2 + E_P^2)} \right]^{\frac{1}{2}} \langle \hat{g}_{ab} \rangle_{k_0}. \quad (6)$$

In particular we will use the scaling properties of the inverse metric,

$$\langle \hat{g}^{ab} \rangle_k = \left[\frac{k^4(k_0^2 + E_P^2)}{k_0^4(k^2 + E_P^2)} \right]^{\frac{1}{2}} \langle \hat{g}^{ab} \rangle_{k_0}. \quad (7)$$

We define the scaling factor in (7), introducing a function $F(k)$: $\langle \hat{g}^{ab} \rangle_k = F(k) \langle \hat{g}^{ab} \rangle_{k_0}$. From the explicit form of $F(k)$ we have three different phases where the behavior of $F(k)$ can be approximated as follows,

$$\begin{aligned} F(k) &\sim 1, \quad k \sim k_0, \\ F(k) &\sim k^2, \quad k_0 \ll k \ll E_P, \\ F(k) &\sim k, \quad k \gg E_P. \end{aligned} \quad (8)$$

We consider $F(k)$ constant for $k \lesssim k_0$; in particular we require that $F(k) \sim 1$, $\forall k \lesssim k_0$. To simplify the calculations without modifying the scaling properties of the metric we introduce the new function $\mathcal{F}(k) = F(k) + 1$. The behavior of \mathcal{F} is exactly the same of (8) but with better properties in the infrared limit useful in the calculations. We define here the scale function $\mathcal{F}(k)$ for future reference in the next sections,

$$\mathcal{F}(k) = \left[\frac{k^4(k_0^2 + E_P^2)}{k_0^4(k^2 + E_P^2)} \right]^{\frac{1}{2}} + 1. \quad (9)$$

We can repeat the scaling analysis of this section in the case of a four dimensional spin-foam model. In this case the area eigenvalues are $A_j = l_P^2 j$ [5] and the scaling of the $4d$ metric is

$$\langle \hat{g}^{\mu\nu} \rangle_k = \frac{k^2}{k_0^2} \langle \hat{g}^{\mu\nu} \rangle_{k_0}, \quad (10)$$

where $\mu, \nu = 1, \dots, 4$. Given the explicit form of the scaling in (10) we introduce the scaling function $\mathbb{F}(k) = k^2/k_0^2 + 1$. The infrared modification introduced by hand does not change the high energy behavior of the scaling

function and we can take $k \in [0, +\infty[$ in the calculations. A different ordering in the area operator quantization can give a different spectrum $A_j = l_P^2(2j+1)$ [5], [6]. The scaling function in this case is $\mathbb{G} = (k^2(k_0^2 + 2E_P^2))/(k_0^2(k^2 + 2E_P^2)) + 1$. Where we have introduced the usual infrared modification: $+1$.

b. The spectral dimension in diffusion processes We recall here the definition of spectral dimension of diffusions processes. Consider the diffusion of a scalar probe particle on a d -dimensional Euclidean manifold with a fixed smooth metric $g_{ab}(x)$. The heat-kernel $K_g(x, x'; T)$ gives the probability for the scalar test particle to diffuse from the point x' to x in a diffusion time T . The heat-kernel satisfies the following heat-equation

$$\partial_T K_g(x, x'; T) = \Delta_g K_g(x, x'; T), \quad (11)$$

where $\Delta_g = g^{-1/2} \partial_a (g^{-1/2} g^{ab} \partial_b \phi)$ is the scalar field Laplacian. The heat-kernel is a matrix element of the operator $\exp(T \Delta_g)$, $K_g(x, x'; T) = \langle x' | \exp(T \Delta_g) | x \rangle$, as we can verify,

$$\begin{aligned} \partial_T K_g(x, x'; T) &= \partial_T \langle x' | e^{T \Delta_g} | x \rangle = \\ &= \int dy \langle x' | \Delta_g | y \rangle \langle y | e^{T \Delta_g} | x \rangle = \Delta_g K_g(x, x'; T). \end{aligned} \quad (12)$$

In the random walk picture the trace per unit volume, $P_g(T) = V_d^{-1} \int d^d x \sqrt{g(x)} K_g(x, x; T) = V_d^{-1} \text{Tr} e^{T \Delta_g}$, is interpreted as an average return probability. For $T \rightarrow 0$, $P_g(T)$ has an asymptotic expansion, $P_g(T) = (4\pi T)^{-d/2} \sum_{n=0}^{+\infty} c_n T^n (4\pi T)^{d/2}$, and for an infinitely flat space $P_g(T) = 1/(4\pi T)^{d/2}$. From $P_g(T)$ we can extract the dimension of the target manifold: $d = -2 \partial \ln P_g(T) / \partial \ln T$.

In quantum geometry and quantum space-time we define the spectral dimension for a d -dimensional manifold in analogy with the classical formula,

$$\mathcal{D}_s = -2 \frac{\partial \ln P_g(T)}{\partial \ln T}. \quad (13)$$

The formula (13) will be our definition of spectral (*fractal*) dimension in both the spatial section and the space-time.

c. Spectral dimension in Quantum Gravity. In this section we calculate the spectral dimension of the spatial section in LQG and of the space-time for spin-foam models [5].

3d Spatial Section. We suppose to have a smooth Riemannian metric at any energy scale k that we denote $\langle g_{ab} \rangle_k$ and we go to probe the space at any scale $0 \lesssim k < +\infty$. As explained in the previous section we must study the properties of the Laplacian operator of a $3d$ manifold. Given the scaling properties of the inverse metric (9) we can deduce the scaling properties of the Laplacian,

$$\Delta(k) = \mathcal{F}(k) \Delta(k_0). \quad (14)$$

We suppose that the diffusion process involves only a small interval of scales where $\mathcal{F}(k)$ does not change much.

Under this assumption the heat-equation must contain $\Delta(k)$ for the specific fixed value of k ,

$$\partial_T K_g(x, x'; T) = \Delta(k) K_g(x, x'; T). \quad (15)$$

We denote the eigenvalues of $\Delta(k_0)$ by $-E_n$ and introducing a resolution of the identity in terms of the eigenvectors of $\Delta(k_0)$, we find the following solution of equation (15),

$$K(x, x'; T) = \sum_n \phi_n(x) \phi_n(x') e^{-T \mathcal{F}(k) E_n} \quad (16)$$

If $\Delta(k_0)$ corresponds to the flat space the eigenfunctions are plane waves, $\phi_n \rightarrow \phi_p \sim \exp(ipx)$, and the eigenvalues of $\Delta(k_0)$ are $-E_n = -p^2$. The eigenfunctions resolve length scales $l \sim 1/p \sim 1/\sqrt{E_n}$. This suggest that when the manifold is probed with mode of eigenvalue E_n it feels the metric $\langle g_{ab} \rangle_k$ for the scale $k = \sqrt{E_n}$. We can calculate the trace of $K(x, x'; T)$ on the plane wave basis,

$$\begin{aligned} P(T) &= V_d^{-1} \int d^d x \langle x | e^{T \Delta(k)} | x \rangle \\ &= V_d^{-1} \int d^d x \langle x | e^{T \mathcal{F}(k) \Delta(k_0)} | x \rangle \\ &= \int \frac{d^d p}{(2\pi)^d} e^{-T \mathcal{F}(k) p^2}. \end{aligned} \quad (17)$$

Where we have introduced the spectrum of the operator $\Delta(k_0)$ and the scaling (14). We have said that a mode of eigenvalue E_n sees a metric $\langle g_{ab} \rangle_k$ for the scale $k = \sqrt{E_n} = p$, then we can identify $p \equiv k$ in (17) obtaining

$$P(T) = \int \frac{d^d k}{(2\pi)^d} e^{-T \mathcal{F}(k) k^2}. \quad (18)$$

We have now all the ingredients to calculate the spectral dimension in LQG. In LQG we will restrict all the previous integrals to the case $d = 3$.

Using the relation (18) and the definition of spectral dimension (13) we have

$$\mathcal{D}_s = 2T \frac{\int d^3 k e^{-k^2 \mathcal{F}(k) T} k^2 \mathcal{F}(k)}{\int d^3 k e^{-k^2 \mathcal{F}(k) T}}. \quad (19)$$

Given the explicit form of the scaling function $\mathcal{F}(k)$ we are not able to calculate an analytical solution. We have calculated the spectral dimension (19) numerically obtaining a function of T which is plotted in Fig.1. From the plot in Fig.1, but also calculating analytically the spectral dimension in the three different regimes of (8), we have

$$\mathcal{D}_s = \begin{cases} 2 & \text{for } k \gg E_P, \\ 1.5 & \text{for } k_0 \ll k \ll E_P, \\ 3 & \text{for } k \lesssim k_0. \end{cases} \quad (20)$$

We can conclude that in LQG we have three different phase that we will try to interpret in the discussion section.

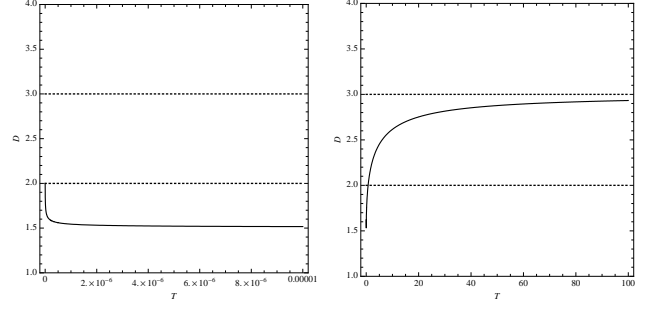


FIG. 1: Plot of the spectral dimension as function of the diffusion time T . We can see three different phase from the left to the right. The first plot is

4d Space-Time. Using the scaling property of the space-time metric in (10) we can calculate the spectral dimension of the 4d-manifold. We use the notation \mathbb{D}_s for the space-time spectral dimension,

$$\mathbb{D}_s = 2T \frac{\int d^4 k e^{-k^2 \mathbb{F}(k) T} k^2 \mathbb{F}(k)}{\int d^4 k e^{-k^2 \mathbb{F}(k) T}}, \quad (21)$$

where $\mathbb{F}(k)$ is the scaling of the metric given in (10). In Fig.2 is given a plot of the spectral dimension as function of the diffusion time T . For $T \rightarrow 0$ (or $k \sim E_P$) the we obtain spectral dimension $\mathbb{D}_s = 2$ and for $T \rightarrow \infty$ (or $k \rightarrow 0$) we obtain $\mathbb{D}_s = 2$. We can consider the high and low energy limit obtaining the following behavior of the spectral dimension,

$$\mathbb{D}_s = \begin{cases} 2 & \text{for } k \gtrsim E_P, \\ 4 & \text{for } k \ll E_P. \end{cases} \quad (22)$$

Our result in space-time is in perfect accord with the results in CDT & ASQG [1], [2]. If we use the scaling function $\mathbb{G}(k)$ defined at the end of the section *a*. one we obtain the same behavior of the spectral dimension in the case we consider the ultraviolet cutoff $k < E_P$ (this cutoff is suggested by the area spectrum $A_j = l_P^2(2j+1)$ which contains the +1 gap on the area eigenvalue for $j = 0$). If we consider the possibility the momentum $k \geq E_P$, we obtain the spectral dimension $\mathbb{D}_s = 4$ for $T \rightarrow 0$ (or $k \rightarrow +\infty$). The behavior of \mathbb{D}_s is instead the same of (22) for $k < E_P$. This high energy behavior of the spectral dimension is interesting if we consider the space-time Ricci invariant $R(g) = R_\mu^\mu(g)$. Under the rescaling $\mathbb{G}(k)$ the Ricci curvature scales as: $R(g)_k = \mathbb{G}(k) R(g)_{k_0}$. At short distances or $k \rightarrow +\infty$ $R(g)_k$ is upper bounded as it is manifest considering the limit: $\lim_{k \rightarrow \infty} \mathbb{G}(k) \sim (E_P/k_0)^2$. The upper bound of the curvature could be a sign of singularity problem resolution showed in cosmology and black holes in the minisuperspace simplification of quantum gravity [9]. We conclude the section considering the case the area spectrum is $A_j = l_P^2 \sqrt{j(j+1)}$. In this case the scaling function is the same given in (9) but the momentum k is now four

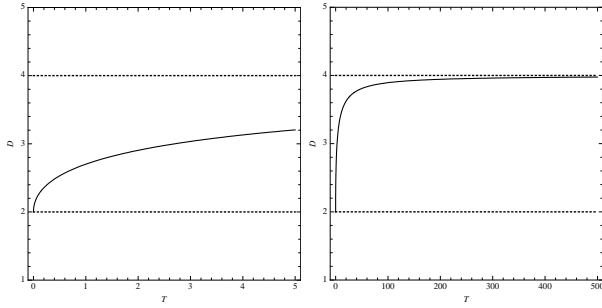


FIG. 2: Plot of the space-time spectral dimension \mathbb{D}_s . We have an high energy phase of spectral dimension $\mathbb{D}_s = 2$ and a the $4d$ low energy dimension.

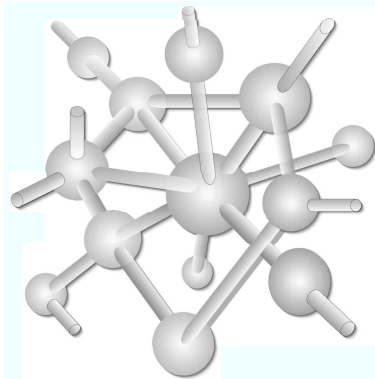


FIG. 3: This artistic picture represents the possible interpretation of the spectral dimension given in the discussion section of the paper. The tiny connections in the picture have a regular distribution in the $3d$ manifold but the 1.5 dimension obtained in the paper suggest a fractal phase around the Planck scale. The pictures focuses on the regime when $\mathcal{D}_s \lesssim 2$.

dimensional. The spectral dimension has the same behavior plotted in Fig.2 in the case $k < E_P$ and instead $\mathbb{D}_s = 8/3$ in the trans-planckian limit ($k \gg E_P$). However if we do not consider the trans-Planckian limit we obtain the same spectral dimension (22) for any form of the area spectrum considered in this section.

Conclusions and Discussion. In this paper we have calculated explicitly the spectral dimension (\mathcal{D}_s) of the spatial section in LQG using the area spectrum scaling and also some strong hipotesys. We have obtained \mathcal{D}_s as

function of the diffusion time T or equivalently as function of the length scale. We have three phases: a short scale phase $l \ll l_P$ of spectral dimension $\mathcal{D}_s = 2$, an intermediate scale phase $l_P \ll l \ll l_0$ of spectral dimension $\mathcal{D}_s = 1.5$ and a large scale phase of $\mathcal{D}_s = 3$. We have calculated the spectral dimension for the space-time in the contest of spin-foam models and we have obtained $\mathbb{D}_s = 2$ at the Planck scale and $\mathbb{D}_s = 4$ at low energy. This result is same obtained in CDT & ASQG [1], [2]. A different area spectrum that come from a different quantum ordering [5] gives the same result until the Planck scale but a new a different behavior in the trans-Planckian regime.

We give now a possible interpretation of the spectral dimension for the spatial section. We can interpret the running of the spectral dimension in the following way. First of all we want to underline that the probe scalar field is just a fictitious field and not a physical scalar field. Consider a scalar field of weave length $\lambda \ll l_P$, at this energy the probe field feels a $2d$ manifold, or in other words it feels the boundary Planck area of a three dimensional chunk of space, that we can imagine of zero volume. The field propagates on a the $2d$ boundary of the chunk of space until it feels tiny wormholes that connects different chunks of space. In this phase (we are in the regime $\lambda \gtrsim l_P$) the field propagates across the wormholes feeling an effective 1.5 -dimensional manifold. The initial chunk is connected to many others chunks then when $\lambda \gg l_P$ it feels a $3d$ manifold. We have assumed the connections of neighbouring chunks are standard wormholes but instead, because the spectral dimension 1.5 , the connections define a fractal structure. In other words we can say that *the probe scalar field (for $\lambda \lesssim l_P$) feels a fractal multi-connected manifold of $\mathcal{D}_s \lesssim 2$* . The pictures in Fig.(3) represents *artistically* our interpretation. If we think in terms of the dual Hilbert space of spin-networks states the wormholes can be interpret as the two dimensional blowing up of the edges in the spin-network graph.

Our result is consistent with the LQG interpretation [4]. In LQG, on any spatial section, there is a minimum non zero area eigenvalue that we can consider (in our interpretation) the manifold where effectively the fictitious scalar field diffuses at high energy.

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